Category Theory for Modular Design: An IoT Example

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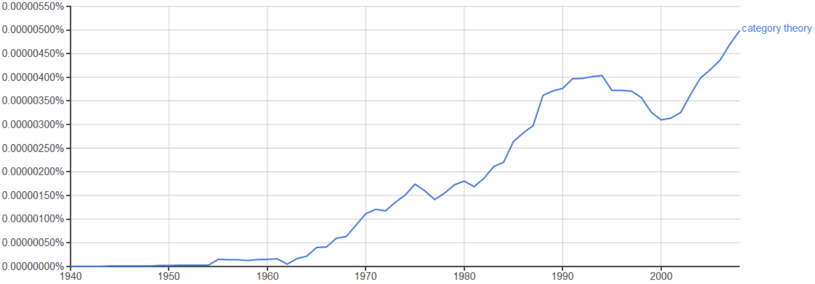
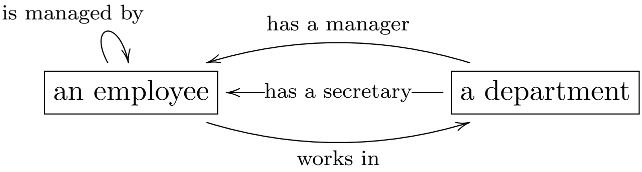
Category Theory



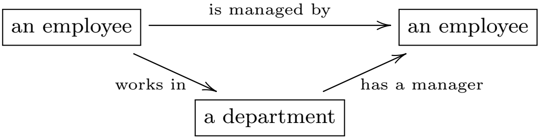
- Developed in the 1940's to connect different branches of mathematics.
- Reveals deep connections between formal logic, computer science and theoretical physics.
- Mathematical study of (implementation-independent) structure.

What is a category?

- Directed graph + path equivalence



- "Every employee is managed by the manager of his department."



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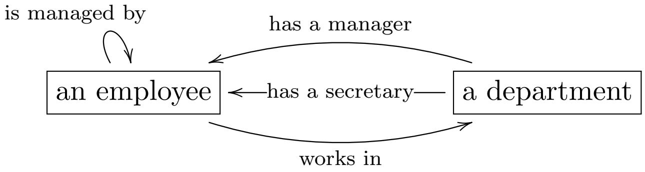
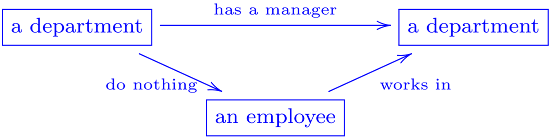


- "Every manager works in the same department she manages."



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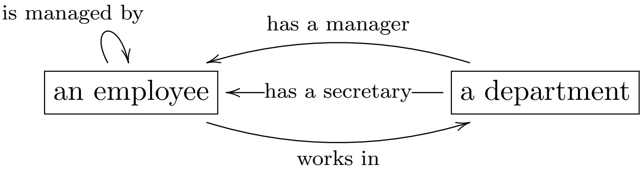
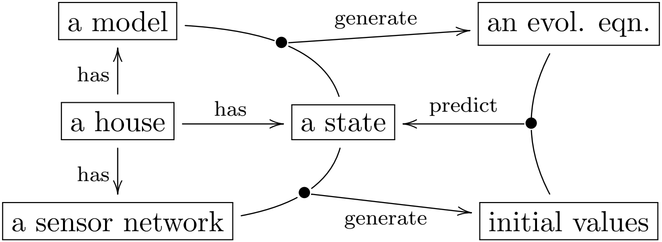


- Intuitively, each box is a set,
  each arrow is a function.
  each path is a composition of functions,
  each equivalence is an equation between composites.

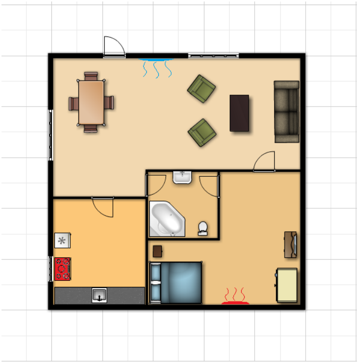
Plan

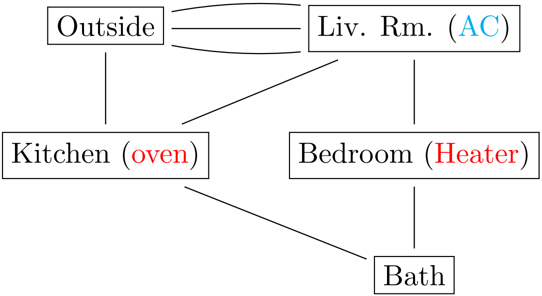
A high-level interface



- This is a *formal* specification of an interface.
- Fill in the details piece by piece, then integrate.

A simple house model





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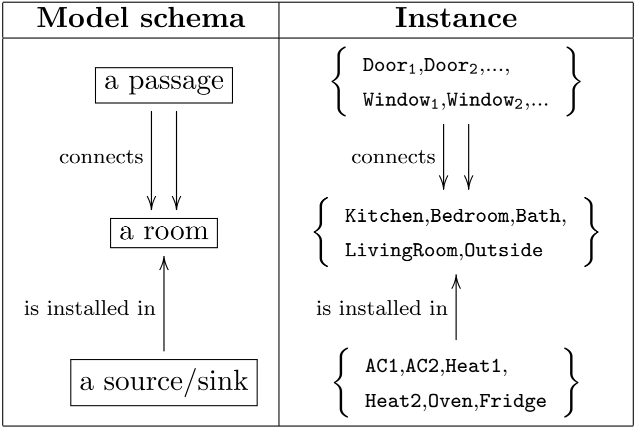
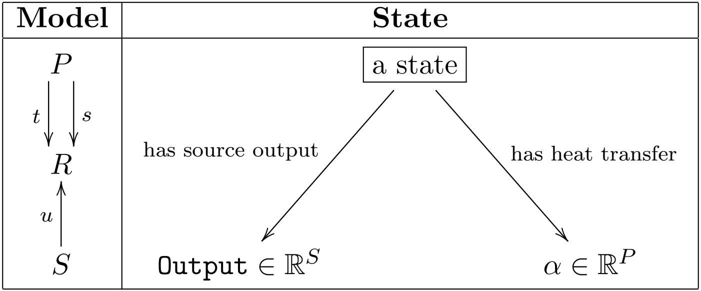


- Categorical models automatically provide database schemas.

Plan

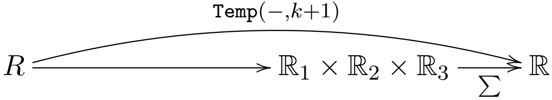
Model + State $\mapsto$ Evolution Equation



$$\texttt{Temp}(r_i, k+1) = \texttt{Temp}(r_i, k) + \sum_{u(s)=r_i} \texttt{Output}(s) + \sum_{t(p)=r_i} \alpha_p \Big(\texttt{Temp}(s(p), k) - \texttt{Temp}(r_i, k)\Big)$$

Modelling the evolution equation

$$\begin{aligned}\texttt{Temp}(r_i, k+1) &= \texttt{Temp}(r_i, k) + \sum_{u(s)=r_i} \texttt{Output}(s) \\ &+ \sum_{t(p)=r_i} \alpha_p \Big(\texttt{Temp}(s(p), k) - \texttt{Temp}(r_i, k) \Big)\end{aligned}$$



Modelling the evolution equation

$R \xrightarrow{\texttt{Temp}(-,k+1)} \mathbb{R}$; $R \longrightarrow \mathbb{R}_1 \times \mathbb{R}_2 \times \mathbb{R}_3 \xrightarrow[\Sigma]{} \mathbb{R}$	Break down sum.
$R \xrightarrow{\texttt{Temp}(-,\texttt{k})} \mathbb{R}_1$	Boundary value.

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$R \xrightarrow{\texttt{curry}_u(\texttt{Output})} \prod_{u(s)=r} \mathbb{R} \xrightarrow{\Sigma} \mathbb{R}_2$	Curry, then sum.

Modelling the evolution equation

$R \longrightarrow \mathbb{R}_1 \times \mathbb{R}_2 \times \mathbb{R}_3 \xrightarrow[\sum]{} \mathbb{R}$, with $R \xrightarrow{\texttt{Temp}(-,k+1)} \mathbb{R}$	Break down sum.
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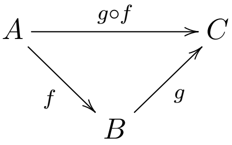
$R \xrightarrow{\texttt{curry}_u(\texttt{Output})} \prod_{u(s)=r} \mathbb{R} \xrightarrow{\sum} \mathbb{R}_2$	Curry, then sum.
$R \xrightarrow{\texttt{curry}_t\left(\alpha_p(\texttt{Temp}(t,k)-\texttt{Temp}(s,k))\right)} \prod_{t(p)=r} \mathbb{R} \xrightarrow{\sum} \mathbb{R}_3$	BVs, algebra, curry, sum.

Plan

Semantic categories

Categories via composition:

- A category consists of *objects* $(A, B, \ldots)$ and *arrows* $(f : A \to B)$.
- Arrows which match "tip-to-tail" can be composed:



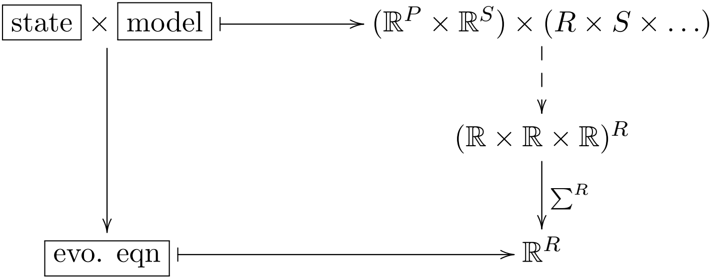
Example: **Set** has sets for objects and functions for arrows.

Maps between categories

A *functor* is a map between categories. This are just like maps between graphs, with one important difference:

Nodes map to nodes, edges map to *paths of edges*.

In the previous example,



Semantic Interpretation

The semantics for categorical models generalize the set-theoretic semantics of first-order logic.

- A "theory" is a (small) syntactic category **Syn**.
- Semantics "occur" in the category **Set**.
- A "model" or interpretation of the theory is a functor

$$\mathbf{Syn} \xrightarrow{\quad I \quad} \mathbf{Set}\ .$$

Semantic Interpretation

The semantics for categorical models generalize the set-theoretic semantics of first-order logic.

- A "theory" is a (small) syntactic category **Syn**.
- Semantics "occur" in some (large) category **Smtc**.
- A "model" or interpretation of the theory is a functor

$$\mathbf{Syn} \xrightarrow{\quad I \quad} \mathbf{Smtc}\ .$$

- The same set-up applies to more exotic semantics.

Sensors + State $\mapsto$ Initial Values

$$\boxed{\text{a state}} \xrightarrow{\text{has sensor locations}} \texttt{loc} \in R^Q$$

$$\boxed{\text{a sensor network}} \xrightarrow{\text{has sensor values}} \texttt{read} \in \mathbb{R}^Q$$

From this data we must produce an initial value map $\texttt{Temp} : R \to \mathbb{R}$.

But what is a map?

Situation: Thermostat in every room.

Semantics: $\mathbf{Syn} \longrightarrow \mathbf{Set}$.

Map: $\texttt{Temp} : R \underset{\texttt{loc}}{\overset{\texttt{loc}^{-1}}{\rightleftarrows}} Q \xrightarrow{\texttt{read}} \mathbb{R}$.

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But what is a map?

Situation: Readings over time.

Semantics: $\mathbf{Syn} \longrightarrow \mathbf{Set}^T$.

Map: $\texttt{Temp} : R \times T \longrightarrow \mathbb{R}$.

Sensors + State $\mapsto$ Initial Values

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From this data we must produce an initial value map $\texttt{Temp} : R \to \mathbb{R}$.

But what is a map?

Situation: Many sensors, or few.

Semantics: $\textbf{Syn} \longrightarrow \textbf{PrLang}$.

Map:

$$\texttt{Temp}(r) = \texttt{if has\_sensors}(r):$$
$$\texttt{return avg}(\texttt{read}(\texttt{sensors}(r)))$$
$$\texttt{else}: \texttt{return avg}(\texttt{read}(\texttt{all\_sensors}))$$

Sensors + State $\mapsto$ Initial Values

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From this data we must produce an initial value map $\texttt{Temp} : R \to \mathbb{R}$.

But what is a map?

Situation: Unreliable sensors.

Semantics: $\mathbf{Syn} \longrightarrow \mathbf{Prob}$.

Map: $\texttt{Temp} : R \longrightarrow \mathbf{ProbDist}(\mathbb{R})$.

Plan

Integration via Pushouts

Formal tools called *colimits* allow us to automatically integrate high-level and low-level models.

1) Begin with two (or more) models.	M_1 M_2

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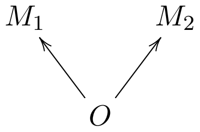
3) Map the overlap into each piece.



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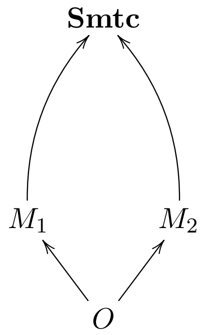
4) Aggregates are defined semantically.



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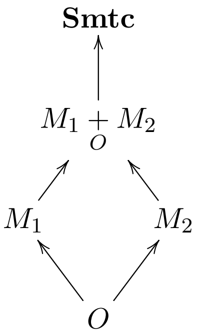
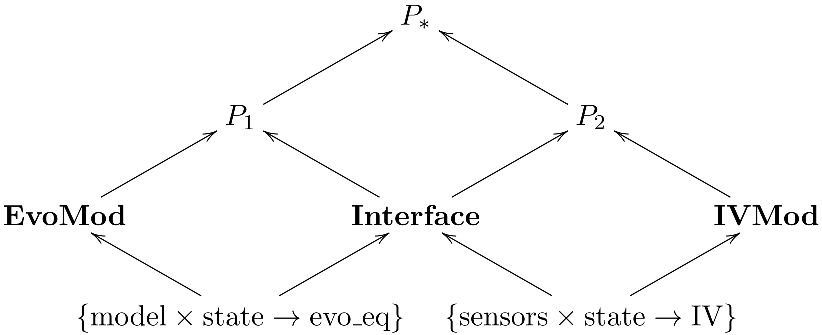
5) Push out.



Iterating Pushouts

Theorems on colimits guarantee correctness for iterated constructions:



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(Some) Advantages of CT

- Extensibility.
- Modularity.
- Close connections with formal logic and programming languages.
- Automatic database integration.
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- Modularity.
- Close connections with formal logic and programming languages.
- Automatic database integration.
- Rich semantics.
- Tools for studying semantic relationships (natural transformations).
- Tools for studying "the same" data in different contexts (adjoints).
- Tools for modelling effects in a functional context (monads).
- Automatic methods for data migration (Kan extension).

Ready for Primetime?

The mathematics is solid; the software is not (yet).

Haskell - A functional programming language using monads.

FQL - A query language and IDE for building categorical databases.

OPL - A graphical programming language based on operads.

DOL - Distributed Ontology Language based on institutions.

No general purpose environments for implementing and analysing categorical models.

Reaching Critical Mass

New developments:

Ologs	-	A human-readable graphical specification for categories.
Spivak	-	*Category Theory for the Sciences*

Want to learn more, or help?

Email: sub@cmu.edu

spencer.breiner@nist.gov

Web: Categorical data project